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1983 J. Phys. A: Math. Gen. 16 2121

(http://iopscience.iop.org/0305-4470/16/10/008)

J. Phys. A: Math. Gen. 16 (1983) 2121–2131. Printed in Great Britain

Further polynomial-type eigenfunctions

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Received 12 November 1982

Abstract. The method developed previously by the author for obtaining eigenfunctions in the form (polynomial) × exponential (polynomial) for a linear second-order differential equation in normal form is extended to embrace the general case when the interaction potential (or square of the refractive index profile) has the form (polynomial)/(polynomial). Various situations arise in the development of the theory, some of which can be concluded algebraically, but the majority require numerical calculations regarding the rank of quite a general matrix containing many unknown parameters.

1. Introduction

In recent years, considerable interest has been focused on the Schrödinger equation with various forms of potential interaction terms. Some of the investigations have embraced numerical methods, while others have been concerned with the production of special eigenfunctions of the form (polynomial) × exp(polynomial), a form that we shall describe as PEP. In particular, we may single out two equations for comment, namely

$$d^2w/dz^2 + [E - z^2 - \lambda z^2/(1+gz^2)]w = 0, \tag{1}$$

$$d^2w/dz^2 + (E - z^2 - \lambda z^{2m})w = 0. \tag{2}$$

Kaushal (1979), Mitra (1978) and Bessis and Bessis (1980) have calculated a wide range of eigenvalues of equation (1), while Biswas *et al* (1973) have considered equation (2) numerically.

On the analytical side, Flessas (1981) has considered briefly the even parity eigenfunctions of (1) in simple cases, with no investigation of the properties of the eigenfunctions. Varma (1981) has also considered briefly both the even parity and the odd parity solutions of this same equation, but again no properties of the eigenfunctions were investigated. Lai and Lin (1982) also gave the odd parity eigenfunctions of (1), and generally expanded the energy eigenvalue E_n as a series in h up to $O(h^4)$, where $h = q/2(1+\lambda)^{1/2}$, $q = 2n+1$. Extensive calculations are reported of the first four energy values for $g = 0.1$, 0.5, 1.0, 2.0 and $\lambda = 0.1$, 0.5, 1.0, 2.0, 5.0, 10.0, 50.0, 100.0. Flessas (1982) extended his work to investigate further eigenvalues and eigenfunctions of (1) by means of definite integrals. Heading (1982) gave a complete theory of these PEP eigenfunctions of (1), together with many properties of the eigenvalues and eigenfunctions. Whitehead *et al* (1982) have also provided a further investigation into solutions of equation (1).

Flessas and Watt (1981) considered a different form of equation (2), using the polynomial potential function $Bz+Cz^2+Dz^3+Ez^4$, $C>0$, $E>0$, where there were two special relations between the parameters B, C, D, E. Saxena and Varma (1982) have investigated the eigenvalues and eigenfunctions for the potential $-z^{-1}+2\lambda z+2\lambda^2z^2$ (our notation) for certain definite values of λ, when the eigenfunctions are restricted to the PEP type. Finally, Znojil (1982) has given a comprehensive investigation for the potential $\Sigma a_r z^r$ (our notation), where r ranges over a set of rational numbers, the a's not being independent when PEP eigenfunctions are required.

In the light of these diverse investigations, we propose here to consider a more general interaction potential that includes the above potentials as simple special cases. We shall seek PEP eigenfunctions of the equation

$$\frac{d^2w}{dz^2}+\left(\frac{\text{polynomial (i) in } z}{\text{polynomial (ii) in } z}\right)w=0, \tag{3}$$

where certain coefficients in the polynomials cannot be arbitrary for PEP eigenfunctions to exist, and where the interval under consideration is $-\infty<z<\infty$ when the denominator in (3) has no zeros. If the denominator has zeros, then PEP eigenfunctions can still be sought in intervals of z that exclude singularities of the equation. A general procedure will be discussed, with certain numerical calculations provided to show what numerical results emerge in special cases. Finally, a brief investigation is made when the two polynomials in the PEP solutions are multiplied by non-integral powers of z, though only integral powers of z are to appear in equation (3). This investigation also enables us to consider solutions of (3) that represent propagating wave forms rather than eigenfunctions (namely, the exponential index will be purely imaginary).

2. The general identity

The PEP function $w(z)=g(z)\,e^{h(z)}$, where g and h are polynomials in z of degrees G and H respectively, is required to satisfy the second-order differential equation in normal form (3). More explicitly, this can be rearranged to the form

$$d^2w/dz^2+[j(z)+p(z)/q(z)]w=0, \tag{4}$$

where j, p, q are polynomials of degrees J, P, Q respectively, where $Q=P+1$ generally after long division. Real zeros of q are excluded from the real range of z under consideration. If the whole of the real-z axis is included, q must be positive definite when z is real so as to avoid singularities.

The second-order differential coefficient of $w=ge^h$ yields

$$w''=[(g''+2g'h')/g+h'^2+h'']w,$$

where a prime denotes differentiation with respect to z. Now write

$$(g''+2g'h')/g\equiv -r-p/q, \tag{5}$$

where r is a polynomial of degree R, yielding finally

$$w''+(r-h'^2-h''+p/q)w=0, \tag{6}$$

in the required form (4).

In order that $w = ge^h$ should satisfy (6), q must be a factor of g so that the denominator q cancels upon substitution into (6). Let $g = qf$, where f is a polynomial of degree F. Identity (5) becomes

$$(qf)'' + 2(qf)'h' + qfr + fp \equiv 0. \tag{7}$$

Polynomials q, f, h, r, p satisfying this identity enable us to write down equations in form (6) possessing PEP solutions.

3. Solution in the general case

The degrees of the four terms in (7) are

$$F + Q - 2, \qquad F + Q + H - 2, \qquad F + Q + R, \qquad F + P,$$

respectively. The two highest values are the second and third, so these must be equal, yielding

$$H = R + 2 \geqslant 2. \tag{8}$$

Now write

$$p = p_0 + p_1 z + \ldots + p_P z^P, \qquad q = q_0 + q_1 z + \ldots + q_Q z^Q,$$

where, for full generality, $Q = P + 1$. Since the polynomials p and q only occur in the ratio p/q, without loss of generality we can write $q_Q = 1$. Also let

$$f = f_0 + f_1 z + \ldots + f_F z^F$$

(where ultimately only the ratios of the f's are relevant),

$$h = h_1 z + h_2 z^2 + \ldots + h_H z^H$$

(where any term h_0 is irrelevant since it appears in an exponential index),

$$r = r_0 + r_1 z + \ldots + r_R z^R,$$

with $R = H - 2$.

The number of coefficients is therefore $2Q + F + 2H$. When the coefficients of the various powers of z in (7) are equated to zero, the number of equations is $F + Q + H - 1$. The overall coefficient of the highest power of z is

$$2f_F q_Q (F + Q) h_H H + f_F q_Q r_R;$$

since this vanishes, we have

$$r_R = -2(F + Q) H h_H. \tag{9}$$

Omitting this equation and coefficient, we have $2Q + F + 2H - 1$ coefficients and $F + Q + H - 2$ equations.

In particular, let the polynomials h and q be specified (though this choice is not necessary). The equations derived from identity (7) by equating coefficients of powers of z to zero all contain $f_0, f_1, \ldots, f_F$ linearly, their coefficients being the unknown coefficients appearing in the polynomials p and r. The $F + Q + H - 2$ equations can be arranged in the form of a matrix equation, namely

$$\boldsymbol{M}\boldsymbol{f} = 0,$$

where the column f contains $f_0, f_1, \ldots, f_F$ in order, and $\boldsymbol{M}$ contains $1+F$ columns and $F+Q+H-2$ rows. For column f to exist, we have the condition that rank $M \leqslant F$. Every minor of order $1+F$ in $\boldsymbol{M}$ must vanish, but in practice this means generally that $Q+H-2$ determinants must vanish.

Since h and q are given and r_R is known, the remaining unknowns are the $P+1$ coefficients of p and the R coefficients of r. Their total, $P+R+1$, equals the number of vanishing determinants, since $Q=P+1$ and $H=R+2$. As soon as p and r have been found, for each set of values of the coefficients the column f follows immediately.

$\boldsymbol{M}$ contains the coefficients of p and r through the terms qfr and fp in (7). Thus p_0 lies on the pseudo-diagonal through the top left-hand corner of $\boldsymbol{M}$; p_1 lies on the next pseudo-diagonal below, and so on. The coefficients of r occur on and to the left of the pseudo-diagonal containing p_0. In particular, $\boldsymbol{M}$ is square when $Q+H=3$, implying that $P+R=0$. Hence $P=R=0$, and $p=p_0$ and $r=r_0$ follows from (9). Thus p_0 is the only outstanding parameter in $\boldsymbol{M}$; in fact, $-p_0$ equals any characteristic root of $\boldsymbol{N}$, where $\boldsymbol{N}$ denotes $\boldsymbol{M}$ with p_0 deleted down its leading diagonal. There will then be $F+1$ characteristic vectors f.

4. Solutions in the case of polynomials in z^m

Identity (7) can be satisfied if all polynomials ascend in powers of z^m, where m is an integer greater than unity. From (8), we deduce that

$$\text{degree of } r = mH-2, \qquad H \geqslant 1,$$

where the degree of h will be mH. Accordingly, we take

$$\begin{aligned}
q &= q_0+q_1 z^m+\ldots+q_Q z^{Qm} \qquad \text{with } q_Q=1,\\
p &= p_1 z^{m-2}+p_2 z^{2m-2}+\ldots+p_Q z^{Qm-2},\\
f &= f_0+f_1 z^m+\ldots+f_F z^{Fm},\\
h &= h_1 z^m+h_2 z^{2m}+\ldots+h_H z^{Hm},\\
r &= r_1 z^{m-2}+r_2 z^{2m-2}+\ldots+r_H z^{mH-2}.
\end{aligned}$$

Other possibilities can be explored, such as when

$$f = f_1 z^k+f_2 z^{m+k}+\ldots f_F z^{Fm+k}, \qquad 0 \leqslant k \leqslant m-1.$$

In the following investigation, we restrict ourselves to the case $k=0$, since the method is the same in every case.

Substitution into identity (7) yields the powers of $z: z^{m-2}$, z^{2m-2}, $\ldots, z^{Fm+Qm+Hm-2}$, implying that $F+Q+H$ equations are obtained. The coefficients of the highest power of z yield

$$2(Q+F)mq_Q f_F m h_H + q_Q f_F r_H = 0$$

or

$$r_H = -2(Q+F)Hm^2 h_H.$$

Apart from this last equation, there will be $Q+F+H-1$ equations, again represented in matrix form as

$$\boldsymbol{M}f = 0,$$

where f denotes the column consisting of $f_0, f_1, \ldots, f_F$. So that rank $\boldsymbol{M} < F+1$, generally it suffices that $Q+H-1$ minors vanish. Again we may postulate that q and h are given, leaving $Q+H-1$ coefficients in p and r to be determined from these equations.

In particular, $\boldsymbol{M}$ is a square matrix when $Q+H=2$, yielding the two cases
(i) $H=1, Q=1$;
(ii) $H=2, Q=0$.
These are examined in the following sections.

5. Case (i) when $\boldsymbol{M}$ is square

With $H=1, Q=1$, let

$$q = q_0 + z^m, \qquad p = p_1 z^{m-2},$$
$$h = h_1 z^m, \qquad r = r_1 z^{m-2},$$
$$f = f_0 + f_1 z^m + \ldots + f_F z^{Fm},$$

where $r_1 = -2(F+1)m^2 h_1$.

To illustrate this when $F=2$, we note that $-p_1$ is a characteristic root of the matrix $\boldsymbol{N}$, where

$$\boldsymbol{N} = \begin{pmatrix} m(m-1)+q_0 r_1 & q_0 m(m-1) & 0 \\ 2h_1 m^2 & 2m(2m-1)+2q_0 h_1 m^2 + q_0 r_1 & 2q_0 m(2m-1) \\ 0 & 4m^2 h_1 + r_1 & 3m(3m-1)+4q_0 h_1 m^2 + q_0 r_1 \end{pmatrix}$$

where $r_1 = -6m^2 h_1$. Thus the differential equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \left(-6m^2 h_1 z^{m-2} - h_1^2 m^2 z^{2m-2} - h_1 m(m-1) z^{m-2} + \frac{p_1 z^{m-2}}{q_0 + z^m}\right) w = 0$$

has a solution

$$w = (q_0 + z^m)(f_0 + f_1 z^m + f_2 z^{2m}) \exp(h_1 z^m);$$

h_1 will be negative if $w \to 0$ as $z \to \pm\infty$ when m is even.

6. Case (ii) when $\boldsymbol{M}$ is square

In identity (7), let $p \equiv 0$, $q = 1$, giving the simpler identity

$$f'' + 2f'h' + fr \equiv 0; \tag{10}$$

this ensures that $w = f\,\mathrm{e}^h$ satisfies the equation

$$\mathrm{d}^2 w/\mathrm{d}z^2 + (r - h'^2 - h'')w = 0,$$

the bracket now being a polynomial, without any denominator occurring.

As a simple example, let $m=2, H=2, h=-z^4$ (so as to achieve a transient solution as $z \to \pm\infty$), $r = r_0 + r_1 z^2$ with $r_1 = 32$. The only cases which can be treated analytically as distinct from numerically occur when $F = 0, 1, 2$. In the latter case, we have

$$\boldsymbol{M} = \begin{pmatrix} r_0 & 2 & 0 \\ 32 & r_0 & 12 \\ 0 & 16 & r_0 \end{pmatrix},$$

with $r_0 = 0, \pm 16$, and the elements of f being $(3, 0, -8)$, $(1, -8, 8)$, $(1, 8, 8)$ respectively, yielding

$$\mathrm{d}^2w/\mathrm{d}z^2 + [r_0 + (r_1 + 12)z^2 - 16z^6]w = 0.$$

The complete range of possibilities for $F = 0, 1, 2$ is given by the following table:

r_0	r_1	f
0	0	1
$4\sqrt{2}$	16	$1, -2\sqrt{2}$
$-4\sqrt{2}$	16	$1, 2\sqrt{2}$
0	32	$3, 0, -8$
16	32	$1, -8, 8$
-16	32	$1, 8, 8$

More generally, let

$$h = -h_1z^m - z^{2m}, \qquad r = r_0z^{m-2} + r_1z^{2m-2},$$
$$f = f_0 + f_1z^m + \ldots + f_Fz^{Fm},$$

where $r_1 = 4Fm^2$. Thus when $F = 3$, we have

$$\boldsymbol{M} = \begin{pmatrix} r_0 & m(m-1) & 0 & 0 \\ 12m^2 & r_0 - 2h_1m^2 & 2m(2m-1) & 0 \\ 0 & 8m^2 & r_0 - 4h_1m^2 & 3m(3m-1) \\ 0 & 0 & 4m^2 & r_0 - 6h_1m^2 \end{pmatrix}.$$

If $\boldsymbol{N}$ denotes the matrix $\boldsymbol{M}$ with the symbol r_0 removed, for general values of F we have

$$N_{jj} = -2(j-1)h_1m^2, \qquad j = 1, 2, \ldots, F+1,$$
$$N_{j,j+1} = jm(jm-1), \qquad j = 1, 2, \ldots, F,$$
$$N_{j+1,j} = 4(F-j+1)m^2, \qquad j = 1, 2, \ldots, F,$$

where $-r_0$ is any characteristic root of $\boldsymbol{N}$. Under these circumstances,

$$w = (f_0 + f_1z^m + \ldots + f_Fz^{Fm})\exp(-h_1z^m - z^{2m})$$

is a solution of the equation

$$\mathrm{d}^2w/\mathrm{d}z^2 + [r_0z^{m-2} + 4Fm^2z^{2m-2} + h_1m(m-1)z^{m-2} + [2m(2m-1) - h_1^2m^2]z^{2m-2}$$
$$- 4h_1m^2z^{3m-2} - 4m^2z^{4m-2}]w = 0. \tag{11}$$

The only case that is susceptible to easy analytical manipulation occurs when $F = 1$. We have

$$N = \begin{pmatrix} 0 & m(m-1) \\ 4m^2 & -2h_1m^2 \end{pmatrix}$$

with

$$r_0^2 - 2h_1m^2r_0 - 4m^3(m-1) = 0$$

and

$$r_0 = h_1m^2 \pm [h_1^2m^4 + 4m^3(m-1)]^{1/2}, \qquad f_0 : f_1 = m(m-1) : -r_0.$$

Computer calculations of the characteristic roots and vectors of N have been made for all combinations given by

$$m = 2, 5, 10, \qquad F = 1, 2, 5, 10,$$
$$h_1 = 0, 0.5, 1, 5, 10, 50, 100,$$

introducing matrices up to order 11. We present a short table of some of the results below, to show the order of magnitudes of the characteristic roots that emerge from the calculations when $m = 2$.

F	$h_1 = 0$	0.5	1	5	10	50	100
1	5.656 85	4	2.928 20	0.784 61	0.398 02	0.079 98	0.040 00
	−5.656 85	−8	−10.928 2	−40.784 6	−80.398 0	−400.080	−800.040
2	1	−2.092 11	8	1.629 44	0.803 92	0.160 03	0.080 00
	−0.113 21	11.455 5	−4.686 29	−37.229 9	−78.461 0	−399.681	−799.840
	0.147 41	−21.363 4	−27.313 7	−84.399 5	−162.343	−800.480	−1600.24
5	66.453 9	−80.056 2	−94.472 2	4.701 20	2.073 87	0.400 56	0.200 07
	−66.453 9	53.787 2	42.195 7	−25.384 2	−72.434 9	−398.480	−799.240
	8.202 63	−15.477 2	−56.467 1	−66.533 8	−152.079	−798.324	−1599.16
	−8.202 63	2.803 36	15.048 1	−114.961	−236.272	−1199.13	−2399.56
	33.415 6	−44.505 3	−24.412 3	−169.250	−324.587	−1600.88	−3200.44
	−33.415 6	23.448 2	−1.892 19	−228.572	−416.701	−2003.59	−4001.80
10	191.353	165.504	−246.743	15.397 1	4.403 75	0.802 73	0.400 34
	−191.353	−218.468	−193.200	−1.658 87	−61.549 1	−396.473	−798.239
	142.580	119.143	141.045	−32.428 5	−133.779	−794.720	−1597.36
	−142.580	−167.311	−143.496	−72.753 7	−211.321	−1193.93	−2396.96
	97.842 1	76.986 2	97.147 0	−119.926	−293.545	−1594.11	−3198.04
	−97.842 1	−77.187 7	−97.999 3	−172.804	−379.999	−1995.24	−3997.60
	57.697 4	−39.443 9	57.710 2	−503.247	−470.339	−2397.32	−4798.65
	−57.697 4	−9.689 55	−57.277 5	−230.662	−564.292	−2800.35	−5600.16
	0	10.708 4	24.170 1	−429.552	−762.179	−3204.33	−6402.16
	23.513 9	−120.070	−22.636 8	−359.382	−661.634	−3609.24	−7204.64
	−23.513 9	39.828 6	1.271 74	−292.983	−865.765	−4015.09	−8007.59

All the roots are real, a fact that can be proved directly from the differential equation (11). As h_1 increases in value, one root remains small, but the others are large negative numbers. Asymptotic forms can be found for the roots, using the same method as given in Heading (1982). This would explain why, for large h_1, the roots for one value of F are repeated with minor changes in value for higher values of F.

7. The special status of the Hermite polynomials

For general values of $m \geqslant 2$, and with $H = 1$, write

$$h = -z^m, \qquad r = r_0 z^{m-2}, \qquad f = f_0 + f_1 z^m + \ldots + f_F z^{Fm};$$

identity (7) yields the matrix

$$M = \begin{pmatrix} r_0 & m(m-1) & 0 & 0 & \ldots \\ 0 & -2m^2 + r_0 & 2m(2m-1) & 0 & \ldots \\ 0 & 0 & -4m^2 + r_0 & 3m(3m-1) & \ldots \\ . & . & . & . & \ldots \end{pmatrix}.$$

In this special case, its determinant can be evaluated immediately,

$$r_0(r_0 - 2m^2)(r_0 - 4m^2) \ldots (r_0 - 2Fm^2).$$

This vanishes when $r_0 = 0, 2m^2, 4m^2, \ldots$.

Moreover, the differential equation satisfied by $w = fz^h$ is

$$\mathrm{d}^2 w/\mathrm{d}z^2 + [r_0 z^{m-2} + m(m-1)z^{m-2} - m^2 z^{2m-2}]w = 0,$$

discussed by Heading (1974, 1977) in connection with phase-integral methods applied to transition points of order greater than unity. Only when $m = 2$ does one term in the bracket equal a constant, the resulting equation being that for the harmonic oscillator, and its solution contains the Hermite polynomials. For any other value of m, the eigenvalue term r_0 multiplies a function of z. The special form of the determinant ensures that in this case an infinite number of PEP solutions are produced.

8. The case when M is not square

To illustrate the nature of the problem, let $m = 2$. The complete resolution of the problem, analytically as distinct from numerically, is only possible when $F = 0, 1$. Thus consider

$$\begin{aligned} Q &= 2, & q &= q_0 + z^4, \\ P &= 1, & p &= p_0 + p_1 z^2, \\ H &= 2, & h &= -z^4, \\ R &= 1, & r &= r_0 + r_1 z^2, \\ F &= 0, 1, & f &= f_0 \text{ or } f_0 + f_1 z^2. \end{aligned}$$

Thus the equation

$$\mathrm{d}^2 w/\mathrm{d}z^2 + [r_0 + r_1 z^2 - 16z^6 + 12z^2 + (p_0 + p_1 z^2)/(q_0 + z^4)]w = 0$$

has a solution

$$w = (f_0 + f_1 z^2)(q_0 + z^4) \exp(-z^4).$$

Substitution into (7) yields five equations. The last gives $r_1 = 48$, and the first four equations may be written in matrix form as

$$\begin{pmatrix} q_0 r_0 + p_0 & 2q_0 \\ 12 + 48q_0 + p_1 & q_0 r_0 + p_0 \\ r_0 & 22 + 32q_0 + p_1 \\ 16 & r_0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = 0, \tag{12}$$

where q_0 is given. When this matrix is made to be of rank 1, we have three equations for r_0, p_0, p_1.

When $F = 0$, we have $r_1 = 32$ and

$$\begin{pmatrix} q_0 r_0 + p_0 \\ 12 + 32q_0 + p_1 \\ r_0 \end{pmatrix} f_0 = 0,$$

yielding $r_0 = p_0 = 0$, and $p_1 = -(12 + 32q_0)$. Hence $w = (q_0 + z^4)\exp(-z^4)$ satisfies the equation

$$d^2w/dz^2 + [44z^2 - 16z^6 - (12 + 32q_0)z^2/(q_0 + z^4)]w = 0.$$

To solve (12) when $F = 1$, write $E = q_0 r_0 + p_0$, $S = 12 + 48q_0 + p_1$, implying that

$$\operatorname{rank} \begin{pmatrix} E & 2q_0 \\ S & E \\ r_0 & 18 - 16q_0 + S \\ 16 & r_0 \end{pmatrix} = 1,$$

giving $E^2 = 2Sq_0$, $E(18 - 16q_0 + S) = 2q_0 r_0$, $Er_0 = 32q_0$. This yields a quadratic equation for S:

$$S^2 + (18 - 16q_0)S - 32q_0 = 0$$

with roots

$$S = 8q_0 - 9 \pm (64q_0^2 - 112q_0 + 81)^{1/2},$$

from which E, r_0, p_0 and p_1 can be found. For higher values of F, numerical procedures are necessary to determine the parameters that make rank $\boldsymbol{M} = F$.

9. Conclusion and further extensions

We have explained a method whereby a solution of the form $g(z)\,e^{h(z)}$ satisfies a second-order differential equation in normal form, the coefficients of w'' and w being polynomials in z. The method involves the calculation of parameters to make the rank of a certain matrix equal to a specific value, or in a special case to ensure that the determinant of a square matrix is zero. The simpler cases can be evaluated algebraically.

But when a fractional index is allowed to precede a polynomial in f and h, though not in the differential equation (4), then a singularity will exist at $z = 0$ in the differential equation.

In the general theory given in § 2, replace f by $z^X f$ where f is a polynomial of degree F, and h by $z^Y h$ where h is a polynomial of degree H; $Y > 0$. The denominator

q will no longer have a constant term q_0. In (6), $r-h'^2-h''$ must be a polynomial j, so condition (7) must be rewritten as

$$(qz^Xf)''+2(q^Xf)'(z^Yh)'+qz^Xf[j+(z^Yh)'^2+(z^Yh)'']+z^Xfp=0. \tag{13}$$

Consider, firstly, $q_0=0, q_1\neq 0$. Then the lowest powers of z in the six terms in (13) are respectively

$$X-1,\quad X+Y-1,\quad X+1,\quad X+2Y-1,\quad X+Y-1,\quad X.$$

The lowest index is $X-1$, and there will be no terms in identity (13) to cancel this unless $Y=0$, which is not a value under discussion. Hence we must have $q_0=q_1=0, q_2\neq 0$, yielding for the lowest indices in the six terms

$$X,\quad X+Y,\quad X+2,\quad X+2Y,\quad X+Y,\quad X.$$

The lowest index X occurs twice, enabling cancellation to take place. So that the fourth term with the lowest index $X+2Y$ can fit into the scheme for identity (13), we must have $2Y=$ integer, and to be relevant, this must be an odd integer for a singularity to exist in z^Yh. We shall write $Y=N/2$, where N is an odd integer.

In (13), the powers of z in the second and fifth terms do not match those in the remaining terms. Hence, for the identity to be satisfied,

$$2(qz^Xf)'(z^Yh)'+qz^Xf(z^Yh)''=0,$$

or

$$(z^Yh)'\propto 1/(qz^Xf)^2.$$

This is clearly not possible when h, q and f are polynomials of more than one term in each; they will be simply powers of z. Accordingly, let $q=z^2$, $h=h_0$, $f=f_0$ (any actual first power of z in f and h being accumulated in z^X and z^Y). Hence

$$(z^{N/2}h_0)'\propto 1/(z^{X+2}f_0)^2,$$

yielding

$$X=-\tfrac{1}{4}N-\tfrac{3}{2}.$$

The remaining identity in (13) is

$$(qz^Xf)''+qz^Xf[j+(z^Yh)'^2]+z^Xfp=0,$$

yielding

$$(X+2)(X+1)+z^2j+z^2h_0^2Y^2z^{2Y-2}+p=0.$$

Hence when $Y=\tfrac{1}{2}$,

$$p=-(X+2)(X+1)-\tfrac{1}{4}h_0^2z,\qquad j=0;$$

when $Y\geqslant\tfrac{3}{2}$,

$$p=-(X+2)(X+1),\qquad j=-h_0^2Y^2z^{2Y-2}.$$

Thus we conclude that $w=z^{-N/4+1/2}\exp(h_0z^{N/2})$ is a PEP solution of

$$w''+[-\tfrac{1}{4}N^2h_0^2z^{N-2}-\tfrac{1}{16}(N^2-4)z^{-2}]w=0 \tag{14}$$

for N odd and greater than 1, and that $w=z^{1/4}\exp(h_0z^{1/2})$ is a PEP solution of

$$w''+\tfrac{1}{16}(3-4h_0^2z)z^{-2}w=0 \tag{15}$$

when $Y=\frac{1}{2}$. Although the details of the derivations are distinct, it should be noted that when $N=1$ in (14), equation (15) is produced. Additionally, N may be an even integer, though in this case the exponential function in the PEP solution does not contain a singularity. The reader should also note how the WKBJ solution of (14) corresponds to the exact solution. The WKBJ solution that ignores the term z^{-2} in (14) is the exact solution of the equation. This is not the same thing as saying that the WKBJ solution is the exact solution of the equation, since the only differential equations with this property are those in which the refractive index is constant or where it is an inverse square function.

Throughout this paper, the PEP solution w has consisted of an exponential function containing a real index. Wave propagation (with a real square of the refractive index profile) would lead to exponential functions with a purely imaginary index. In this case, we may replace h_0 by $\mathrm{i}h_0$, implying that $w=z^{-N/4+1/2}\exp(\mathrm{i}h_0z^{N/2})$ satisfies

$$w''+[\tfrac{1}{4}N^2h_0^2z^{N-2}+\tfrac{1}{16}(N^2-4)z^{-2}]w=0$$

for all integers N. The complex conjugate w^* is also a solution, meaning that the one equation has two PEP solutions. A more detailed examination would show that, if $w=g\,\mathrm{e}^{\mathrm{i}h}$, then $h'\propto 1/g^2$, implying that h and g are powers of z without attached polynomials.

Acknowledgment

The author thanks Dr T C Redshaw for having undertaken the computer calculations reported in this paper.

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